§ Einstein summation convention

- 2 kinds of indices: upper & lower
- Same index appearing BOTH as an upper & lower index in the same term $\Longrightarrow$ Sum over this index

E.g: coordinates in $\mathbb{R}^n$: $x^1, x^2, \dots, x^n$ (upper)

coordinate vector fields in $\mathbb{R}^n$: $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ (lower)

vector fields in $\mathbb{R}^n$: $X = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i} = a^i \frac{\partial}{\partial x^i}$

Einstein summation convention

E.g. $\mathbf{X}(u^1,u^2) : \mathcal{U} \to S \subseteq \mathbb{R}^3$ parametrization

Write $\boxed{\partial_i := \frac{\partial \mathbf{X}}{\partial u^i}}$ coordinate vector fields

Any tangential vector field $X \in \mathfrak{X}(S)$ can be locally expressed as $X = a^i \partial_i = a^k \partial_k$

"dummy index"

1st f.f: $g_{ij} = \langle \partial_i, \partial_j \rangle$

inverse: $(g^{ij}) = (g_{ij})^{-1}$

2nd f.f.: $A_{ij} = \langle \frac{\partial^2 \mathbf{X}}{\partial u^i \partial u^j}, N \rangle$

E.g. $\langle a^i \partial_i, b^j \partial_j \rangle = g_{ij} a^i b^j$

$g^{ik} g_{kj} = \delta_{ij}$

§ Christoffel Symbols

Given a parametrization $\mathbf{X}(u^1, u^2)$ on S

tangent vectors: $\partial_1 = \dfrac{\partial \mathbf{X}}{\partial u^1}$, $\partial_2 = \dfrac{\partial \mathbf{X}}{\partial u^2}$

unit normal: $N = \dfrac{\partial_1 \times \partial_2}{\| \partial_1 \times \partial_2 \|}$

At each point $p \in S$, we have the orthogonal splitting:

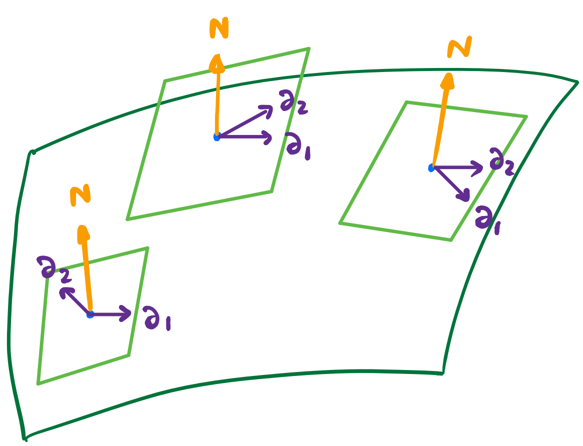
$$\mathbb{R}^3 \cong T_p\mathbb{R}^3 = \underbrace{T_pS}_{} \oplus \underbrace{(T_pS)^\perp}_{}$$

basis: $\boxed{\partial_1, \partial_2} \perp N$

Caution: NOT orthonormal!

$(g_{ij}) = \langle \partial_i, \partial_j \rangle \neq \delta_{ij}$

We have a 2-parameter family of basis of $\mathbb{R}^3$ along S:



Similar to a "moving frame", but NOT orthonormal!

Def$\underline{^n}$: The Christoffel symbols Γ_{ij}^k of a given coordinate system u^1, u^2 is defined as

$$\boxed{\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k}$$

Prop: (1) $\Gamma_{ij}^k = \Gamma_{ji}^k$ (symmetry)

(2) $$\boxed{\Gamma_{ij}^k = \frac{1}{2} g^{k\ell} \left(\partial_i g_{\ell j} + \partial_j g_{i\ell} - \partial_\ell g_{ij} \right)}$$ Koszul formula

Proof: (1) Since ∇ is torsion-free, i.e.

$$\underbrace{\nabla_{\partial_i} \partial_j - \nabla_{\partial_j} \partial_i}_{= (\underbrace{\Gamma_{ij}^k - \Gamma_{ji}^k}_{=0}) \partial_k} = [\partial_i, \partial_j] = 0$$

(2) Since ∇ is metric compatible,

$$X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$$

for any tangential vector fields $X, Y, Z \in \mathfrak{X}(S)$

In particular, if we choose $X = \partial$, $Y = \partial_i$, $Z = \partial_j$

$$\Rightarrow \quad \partial_\ell g_{ij} = \langle T^k_{\ell i}\, \partial_k , \partial_j \rangle + \langle \partial_i , T^k_{\ell j}\, \partial_k \rangle$$

$$= T^k_{\ell i}\, g_{kj} + T^k_{\ell j}\, g_{ik}$$

Cyclicly permuting i, j, ℓ, we obtain using (1)

$$\begin{array}{rl}
- & \partial_\ell g_{ij} = \cancel{T^k_{\ell i}\, g_{kj}} + \cancel{T^k_{\ell j}\, g_{ik}} \\
+ & \partial_i g_{j\ell} = T^k_{ij}\, g_{k\ell} + \cancel{T^k_{i\ell}\, g_{jk}} \\
+ & \partial_j g_{\ell i} = \cancel{T^k_{j\ell}\, g_{ki}} + T^k_{ji}\, g_{\ell k} \\
\hline
 & \partial_i g_{\ell j} + \partial_j g_{i\ell} - \partial_\ell g_{ij} = 2\, g_{k\ell}\, T^k_{ij}
\end{array}$$

Multiplying by $g^{m\ell}$ on both sides

$$g^{m\ell}\left(\partial_i g_{\ell j} + \partial_j g_{i\ell} - \partial_\ell g_{ij}\right) = 2\, \underbrace{g^{m\ell}\, g_{k\ell}}_{=\,\delta^m{}_k}\, T^k_{ij} = 2\, T^m_{ij}$$

Dividing 2 on both sides and switch m to k gives the desired formula.

$\square$

Remark: An important feature of Koszul formula is that we can compute Γ_{ij}^k directly from g_{ij} and its first derivatives, i.e.

$$\boxed{\Gamma_{ij}^k = F(g_{ij}, \partial_\ell g_{ij})}$$

Now, we can derive some sort of "Frenet-type" formulas by differentiating the moving basis $\{\partial_1, \partial_2, N\}$.

Prop:

$$\boxed{\begin{aligned} &\text{(Gauss)} & D_{\partial_i}\partial_j &= \Gamma_{ij}^k \partial_k + A_{ij} N \\ &\text{(Weingarten)} & D_{\partial_i} N &= -g^{jk} A_{ij} \partial_k \end{aligned}} \quad (*)$$

Proof:

$$\begin{aligned} D_{\partial_i}\partial_j &= (D_{\partial_i}\partial_j)^T + (D_{\partial_i}\partial_j)^N \\ &= \nabla_{\partial_i}\partial_j + \langle D_{\partial_i}\partial_j, N\rangle N \\ &= \Gamma_{ij}^k \partial_k + A_{ij} N \end{aligned}$$

This proves the first equation. For the second equation, first notice that $\langle N, N\rangle \equiv 1$, therefore

$$0 = \partial_i \langle N, N\rangle = 2\langle D_{\partial_i} N, N\rangle$$

i.e. $D_{\partial_i} N$ is tangential!

On the other hand, since $\langle N, \partial_j \rangle \equiv 0$,

$$\langle D_{\partial_i} N, \partial_j \rangle = - \langle N, D_{\partial_i} \partial_j \rangle = - A_{ij}$$

So, if we let $D_{\partial_i} N = c_i^{\,k} \partial_k$, then

$$c_i^{\,k} g_{kj} = - A_{ij}$$

$$\Rightarrow \qquad c_i^{\,k} = - g^{kj} A_{ij}$$

This proves the second equation.

$\square$

<u>Remark</u>: From (*) we see that the second f.f A_{ij} can be obtained from the "rate of change of the moving basis $\{\partial_1, \partial_2, N\}$", which in turn gives the Gauss and mean curvatures K and H.

§ Gauss and Codazzi equations

We are now ready to prove one of the most important theorems in classical differential geometry.

Theorema Egregium: (Gauss' Golden Theorem)

The Gauss curvature K is an intrinsic invariant, i.e. it depends only on the 1^{st} f.f. g_{ij} (and its higher derivatives)

The theorem above follows from a set of equations called "constraint equations". To understand these equations, we look at the following:

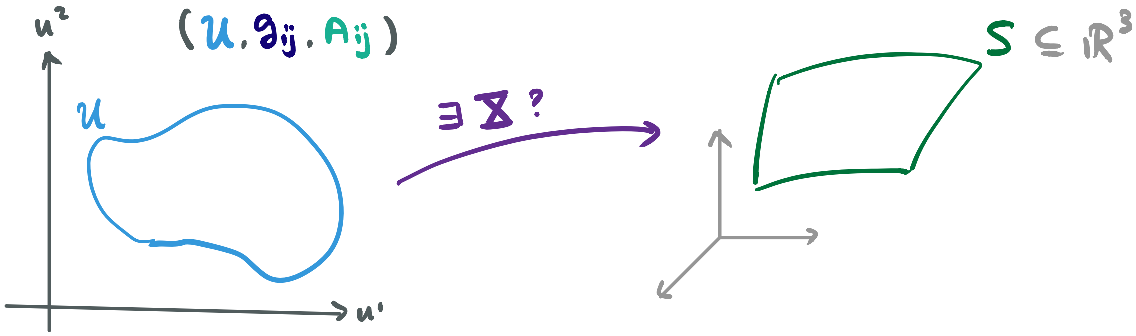
Problem: Given an open set $\mathcal{U} \subseteq \mathbb{R}^2$, and two smooth family of matrices defined on $\mathcal{U}$ s.t.

$$(g_{ij}) \ : \ \text{symmetric \& positive definite}$$

$$(A_{ij}) \ : \ \text{symmetric}$$

Can one find a parametrization $X : \mathcal{U} \to S \subseteq \mathbb{R}^3$ of a surface S s.t.

$$(g_{ij}) = 1^{st} \text{ f.f.} \quad \text{and} \quad (A_{ij}) = 2^{nd} \text{ f.f.} \quad ?$$



<u>Recall</u>: For curves in $\mathbb{R}^2$ or $\mathbb{R}^3$, one can prescribe any curvature and torsion.

The situation is drastically different that the f.f.'s have to satisfy a natural set of "compatibility equations".

<u>Theorem</u>: If (g_{ij}) and (A_{ij}) are the 1st & 2nd f.f. of a surface in $\mathbb{R}^3$ under some local coordinate system, then we have:

Gauss equation:

$$\boxed{\partial_k \Gamma^{\ell}_{ij} - \partial_j \Gamma^{\ell}_{ik} + \Gamma^{p}_{ij} \Gamma^{\ell}_{pk} - \Gamma^{p}_{ik} \Gamma^{\ell}_{pj} = g^{\ell p} (A_{ij} A_{kp} - A_{ik} A_{jp})}$$

Codazzi equation:

$$\boxed{\partial_k A_{ij} - \partial_j A_{ik} + \Gamma^{p}_{ij} A_{pk} - \Gamma^{p}_{ik} A_{pj} = 0}$$

Proof: Let $\mathbf{X}(u^1, u^2): \mathcal{U} \to \mathbb{R}^3$ be a parametrization.

Recall the Gauss & Weingarten equations:

$$\begin{cases} \partial_i \partial_j \mathbf{X} = \Gamma_{ij}^k \partial_k + A_{ij} N \\ \partial_i N = -g^{jk} A_{ij} \partial_k \end{cases}$$

Since partial derivatives of any order commute:

$$\boxed{\partial_k(\partial_i \partial_j \mathbf{X}) = \partial_j(\partial_i \partial_k \mathbf{X})} \quad (\#)$$

We will show that Gauss & Codazzi equations follow from equating the tangential and normal components respectively in $(\#)$.

$$\partial_k(\partial_i \partial_j \mathbf{X}) \overset{\text{Gauss}}{=} \partial_k(\Gamma_{ij}^l \partial_l + A_{ij} N)$$

$$= (\partial_k \Gamma_{ij}^l)\partial_l + \Gamma_{ij}^l(\partial_k \partial_l \mathbf{X}) + (\partial_k A_{ij}) N + A_{ij}(\partial_k N)$$

$$\underset{\text{Weingarten}}{\overset{\text{Gauss}}{=}} (\partial_k \Gamma_{ij}^l)\partial_l + \Gamma_{ij}^l(\Gamma_{kl}^p \partial_p + A_{kl} N) + (\partial_k A_{ij}) N + A_{ij}(-g^{pl} A_{kp} \partial_l)$$

(grouping tangential and normal terms and renaming some dummy indices)

$$\partial_k(\partial_i \partial_j \mathbf{X}) = (\partial_k \Gamma_{ij}^{\ell} + \Gamma_{ij}^{p} \Gamma_{kp}^{\ell} - g^{p\ell} A_{ij} A_{kp})\, \partial_\ell$$
$$+ (\partial_k A_{ij} + \Gamma_{ij}^{\ell} A_{k\ell})\, N$$

Switching the indices k & j, we obtain

$$\partial_j(\partial_i \partial_k \mathbf{X}) = (\partial_j \Gamma_{ik}^{\ell} + \Gamma_{ik}^{p} \Gamma_{jp}^{\ell} - g^{p\ell} A_{ik} A_{jp})\, \partial_\ell$$
$$+ (\partial_j A_{ik} + \Gamma_{ik}^{\ell} A_{j\ell})\, N$$

Equating the tangential part $\Rightarrow$ Gauss equation.

Equating the normal part $\Rightarrow$ Codazzi equation.

$\square$

It turns out that these are all the equations constraining (g_{ij}) and (A_{ij}).

Bonnet's Theorem: Given (g_{ij}) & (A_{ij}) on an open set $\mathcal{U}$ of $\mathbb{R}^2$ satisfying Gauss & Codazzi equations, then

$\exists$ parametrization $\mathbf{X} : \mathcal{U} \to \mathbb{R}^3$

with $(g_{ij}) = 1^{st}$ f.f. & $(A_{ij}) = 2^{nd}$ f.f.

Proof: Omitted.

Proof of Theorema Egregium: (i.e. K is intrinsic.)

Recall: $$\boxed{K = \frac{\det(A_{ij})}{\det(g_{ij})}}$$

It suffices to show that $\det(A_{ij}) = A_{11}A_{22} - A_{12}^2$ is intrinsic (i.e. depends only on g_{ij} and its derivatives)

Take $i=j=1$, $k=l=2$ in Gauss equation:

$$\partial_2 \Gamma_{11}^2 - \partial_1 \Gamma_{12}^2 + \Gamma_{11}^p \Gamma_{p2}^2 - \Gamma_{12}^p \Gamma_{p1}^2 = g^{2p}(A_{11}A_{2p} - A_{12}A_{1p})$$

$$= g^{21}\underbrace{(A_{11}A_{21} - A_{12}A_{11})}_{=0} + g^{22}(A_{11}A_{22} - A_{12}A_{12})$$

$$\Rightarrow \det(A_{ij}) = \underbrace{\frac{1}{g^{22}}\left(\partial_2 \Gamma_{11}^2 - \partial_1 \Gamma_{12}^2 + \Gamma_{11}^p \Gamma_{p2}^2 - \Gamma_{12}^p \Gamma_{p1}^2\right)}_{\text{depends only on } g_{ij} \text{ (up to 2nd derivatives).}}$$

$\square$